TRIANGULATIONS OF CAMBRIAN LATTICES OF TYPE A

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Abstract. Dillworth’s Theorem states that the maximal size of an antichain is equal to the minimal number of chains needed to cover the partially ordered set. We study the Greene-Kleitman partition of c-Cambrian lattices of type A. We partially compute the Greene-Kleitman partition of the Bipartite Cambrian Lattice. Professor Gordana Todorov is my consultant on the project.

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1. Introduction

Dillworth’s Theorem is one of the most groundbreaking discoveries in the field of combinatorics as it describes a universal property for all partially ordered sets (posets). The theorem states that the width of the largest antichain in a poset is equal to the minimum number of chains needed to cover the poset [2]. For now, if we arbitrarily let \( \mu(\pi) \) represent the size of the largest antichain in the poset \( \pi \), and \( |\lambda(\pi)| \) be the minimum number of chains needed to cover a poset \( \pi \), then

\[
|\lambda(\pi)| = \mu(\pi).
\]

The dual of Dillworth’s theorem proven by Mirsky in [6] states that the size of the largest chain is equal to the minimum number of antichains needed to cover the poset (i.e., \( \lambda(\pi) = |\mu(\pi)| \)).

The Greene-Kleitman Theorem provides a continuous transformation between the main and dual versions of Dillworth’s theorem for a poset \( \pi \) [4]. Greene and Kleitman defined two invariants, \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) and \( \mu = (\mu_1, \mu_2, \cdots, \mu_m) \), to describe weakly decreasing chain and antichain
coverings respectively; see the definition in 3.2. The Greene-Kleitman theorem shows that these invariants are actually conjugate partitions and can be written as

\[ \lambda_j = |\{k|\mu_k \geq j\}|, \]
\[ \mu_j = |\{k|\lambda_k \geq j\}|. \]

This conjugacy is clear to see if \(\lambda\) and \(\mu\) are represented with a Young Diagrams; see Figure 4 and Example 3.7. These results in total were very significant to the field of combinatorics as now we have tools to analyze and understand all partially ordered sets in detail.

One poset which is particularly interesting is the Tamari Lattice. Introduced by Tamari in [1], the most famous example of the Tamari Lattice is a partially ordered set where the elements represent binary bracketing of \(n\) objects. But the Tamari lattice has other realizations. This is because the number of elements for each \(n\) is the \(n\)-th Catalan number, and the Catalan number has over one hundred realizations [10]. The most relevant realization for this paper is of triangulations of \((n + 2)\)-gons that look as in Figure 5.

In the past, some people have tried to apply the Greene-Kleitman invariant \(\lambda\) to the Tamari Lattice. In Early’s PhD thesis, he has found the first three parts of the invariant for the Tamari Lattice [3]. Lim and Zhang followed suite and found the fourth and fifth parts of the Greene-Kleitman invariant for the Tamari Lattice [7].

Reading [9] generalized these Tamari triangulations of polygons to adjust for any \((n + 2)\)-gon of choice (i.e., triangulations which happen if one adjusts the positions of vertices in the polygon). He calls these new posets Cambrian Lattice. This paper is interested in both the Tamari Lattice as well as one other of these Cambrian lattices called the Bipartite Cambrian Lattice. Here, the polygon has vertices as described in 3.14. For an example of the Bipartite Cambrian Lattice, see Figure 1.

The problem of this paper is to find as many parts of the Greene-Kleitman invariant as possible for both the Bipartite Cambrian Lattice and Tamari Lattice. This paper also seeks to find an understanding of these concepts from a representation-theoretic perspective as the triangulations of the polygons represents quivers of type \(A\). Applications to this result would be contributing tools and techniques to other mathematicians for finding parts of the Greene-Kleitman invariant for the same or other posets. Also, another application is to further bridge the connection between the fields of algebraic combinatorics and representation theory and develop parallel meaning of objects in one
field to the other. In this paper, we introduce the following theorem which finds the first \( \lfloor \frac{n-1}{2} \rfloor \) parts of the Greene-Kleitman invariant.

**Theorem 1.1 (Bipartite Theorem).** Let \( n \geq 4 \). In the \( s_1 s_3 \ldots s_2 s_4 \ldots \) -Cambrian lattice and in the \( s_2 s_4 \ldots s_1 s_3 \ldots \) -Cambrian lattice, we have

\[
\lambda_1 - 2 = \lambda_2 = \lambda_3 = \ldots = \lambda_{\lfloor \frac{n-1}{2} \rfloor} > \lambda_{\lfloor \frac{n-1}{2} \rfloor + 1}.
\]

The rest of the paper is organized as follows. In Section 2, we outline the problem being examined in this paper. In Section 3, we review some necessary mathematics needed to understand the problem statement in its entirety. In Section 4, we define commutation classes of \( c \)-sortable words and maximum length chains. Finally, in Section 5, we prove Theorem 1.1.

### 2. Problem Statement

The first goal of this paper is to find a lowerbound for the size of the largest antichain for the Tamari lattice. In other words, what is the minimum size of \( \mu_1 \) for the Tamari Lattice for all \( n \)? In the past, researchers such as Early and Zhang [7] have found the first five parts of \( \lambda \) and have conjectured about finding more parts; see Figure 2. However, the width of the poset is not something which is known. This has been a challenging problem to solve, and the only well-known techniques for solving the largest antichain come from trying to find the largest antichain in the right-weak order of permutations.

A second goal of this paper is to find as many parts of \( \lambda \) for the Bipartite Cambrian Lattice for all \( n \) as possible. Finding parts of \( \lambda \) has not yet been extended into the Bipartite Cambrian Lattice, so this research would glean information in new mathematical territory. In order to find the parts of \( \lambda \), it must be determined what these lattices look like for all \( n \), something which has not been done before. It would be nice to have results similar to the ones in Figure 2.

The last goal is to study the Greene-Kleitman partition of Tamari lattices and other Cambrian lattices of type A, but from the approach from representation theory. In particular, there might be a connection between a special subposet in the Bipartite Cambrian Lattice and the Coxeter group. The subposet represents elements which might be in the longest chains of the Bipartite Lattice used to find the first several parts of \( \lambda \), and the Coxeter group helps to uncover this subposet. For example, this subposet would be similar to the subposet represented by bold lines in Figure 2. If this connection is made, this might aid in understanding how the subposet works and if it can give any information to the largest parts of \( \lambda \).

### 3. Necessary Mathematics

#### 3.1. Greene-Kleitman Theorem

In this section, we review the necessary mathematics needed to understand the Greene-Kleitman Theorem which is directly related to the research problem.

**Definition 3.1.** A partially ordered set \( S \) is a set equipped with a binary relation \( < \). For \( x, y \in S \), we use \( x < y \) to signify that \( x \) precedes \( y \).

A chain in \( S \) is a sequence of elements \( x_1, x_2, \ldots \) such that \( x_1 < x_2 < \cdots \). An antichain in \( S \) is a subset \( A \) of \( S \) of elements such that every two distinct \( x, y \in A \) are incomparable (i.e., neither \( x < y \) nor \( y < x \)).
Definition 3.2. A partition of \( n \in \mathbb{N} \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) of integers \( \lambda_k \geq 0 \) where the \( \lambda_k \) are weakly decreasing, that is,

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l
\]

and \( \lambda_1 + \lambda_2 + \cdots + \lambda_l = n \). The Greene-Kleitman Invariants are partitions, where \( n \) is the number of elements in the poset. The terms \( \lambda_1, \lambda_2, \ldots, \lambda_l \) are called parts. If \( \lambda \) is a partition of \( n \), we write \( |\lambda| = n \).

With these definitions, we now introduce the Greene-Kleitman Theorem [4]:

**Theorem 3.3** (Green-Kleitman Theorem). Let \( A_j \) denote the maximum size of \( j \) disjoint chains. Let \( B_j \) denote the maximum size of \( j \) disjoint antichains. Then \( A_j \) is equal to the number of \( k \) for which \( B_k \geq j \), and \( B_j \) is equal to the number of \( k \) for which \( A_k \geq j \).

Using the notation from Theorem 3.3, let \( \lambda_k = A_k - A_{k-1} \) where \( A_0 = 0 \), and let \( \mu_k = B_k - B_{k-1} \).

We define the **Greene-Kleitman invariants** \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \).
Example 3.4. The poset in Figure 3 has its largest union of one chain (its largest chain) from \{a, b, d, f\} and its largest union of two disjoint chains from \{a, b, d, f\} and \{c, e\}. Therefore, \(A_1 = 4\) and \(A_2 = 6\). Computing \(\lambda\), we have \(\lambda = (4 - 0, 6 - 4) = (4, 2)\).

This poset has its largest union of one antichain (its largest antichain) from \{b, c\}, its largest union of two disjoint antichains from \{b, c\} and \{d, e\}, its largest union of three disjoint antichains from \{b, c\}, \{d, e\}, and \{f\}, and finally its largest union of four disjoint antichains from \{b, c\}, \{d, e\}, \{f\}, and \{a\}. Therefore, \(B_1 = 2, B_2 = 4, B_3 = 5, \) and \(B_4 = 6\). Therefore, \(\mu = (2 - 0, 4 - 2, 5 - 4, 6 - 5) = (2, 2, 1, 1)\)

Remark 3.5. In Example 3.4, it so happens that chains were chosen such that the set of the largest union of \(k\) chains was a subset of the set of the largest union of \(k + 1\) chains. This is also true for the antichains in the example. However, this is not always the case for all posets. Likewise, there is not one unique choice of chains and antichains which compose the largest union. For example, one could choose to compose \(A_1\) from the elements \{a, d, e, f\}.

Definition 3.6. The **Young Diagram** of a partition \(\lambda\) is an array of boxes having \(k\) left-justified rows with row \(i\), where \(k\) is the number of parts of \(\lambda\), containing \(\lambda_i\) boxes for \(1 \leq i \leq k\).

The **conjugate partition** of \(\lambda\) is the partition \(\mu\) whose shape is obtained from the Young Diagram of \(\lambda\) by interchanging rows and columns. Equivalently, the Young Diagram of \(\mu\) is the reflection of the Young Diagram of \(\lambda\) about its main diagonal \(y = -x\).

We now introduce an example which is helpful in understanding partially ordered sets, \(\lambda, \mu\), and their relationship to each other:

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & & \\
& & \bullet & \\
& & & \bullet
\end{array}
\]

\[
\begin{array}{cccc}
\bullet & \bullet & & \\
& & \bullet & \\
& & & \bullet
\end{array}
\]

**Figure 4.** Young diagrams for \(\lambda = \{4, 2\}\) and its conjugate partition \(\mu = \{2, 2, 1, 1\}\)

Example 3.7. Suppose we have \(\lambda = (4, 2)\) and its conjugate \(\mu = (2, 2, 1, 1)\) given by a poset such as in Figure 3. In Figure 4, we can represent \(\lambda\) and \(\mu\) respectively with a Young Diagram. We
can use the Young Diagram to check the different parts of $\lambda$ and $\mu$. As an example, we see that $\lambda_2 = 3 = |\{k | \mu_k \geq j\}|$.

3.2. Triangulations of polygons. Studying the values of $A_j$ and $B_j$ from Theorem 3.3 for specific posets is an especially interesting problem. In this section, we define two posets of special interest: the Tamari Lattice and the Bipartite Cambrian Lattice.

Definition 3.8. Take $Q$ to be a convex $(n + 2)$-gon with vertices labeled $0, 1, \cdots, n, n + 1$. We put the 0 and $n + 1$ vertices horizontally across from each other and connect them with a line. We put the remaining vertices between them, below the line, carving out a boat-like shape. We add the condition that these new vertices increase in value from left to right. This is the outline of the Tamari Polygon.

A triangulation of a polygon $Q$ is a tiling of $Q$ by triangles whose vertices are contained in the vertex set of $Q$ [9].

![Figure 5. One Tamari Polygon for $n = 7$](image)

**Example 3.9.** In Figure 5, we can see one example of a triangulation of a Tamari Polygon with $n = 7$. As we will later learn, these triangulations will make up the elements for the Tamari Lattice.

Definition 3.10. Let $X$ denote the set of triangulations of an $n$-gon with vertex labels $1, 2, \ldots, n$. For any $x \in X$, a diagonal flip takes an edge $(i, j)$ in the triangulation $x$, removes the edge to reveal a quadrilateral, and rearranges the edge to be a new edge $(k, l)$ where $k$ and $l$ are the opposite corners of the quadrilateral. We call the edge involved in the diagonal flip the “flipped edge”. A diagonal flip should preserve the fact that $x$ is a triangulation. Two triangulations $x$ and $y$ which differ by one diagonal flip retain $x < y$ if the slope of the flipped edge in $y$ is more positive than the slope of the flipped edge in $x$. Two triangulations $x, z$ satisfy $x < z$ if $z$ can be reached from $x$ by a sequence of edge flips in an increasing order.

**Example 3.11.** In Figure 6, we see that the edge $(3, 6)$ has a negative slope, and gets "flipped" to $(2, 7)$ which has a positive slope.

Definition 3.12. The elements of the Tamari Lattice for a given $n$ are all of the possible triangulations of this Tamari Polygon. A relationship between elements in the Tamari Lattice is given by diagonal flips.

Definition 3.13. A Hasse Diagram is a directed graph used to diagram posets. The nodes of the Hasse Diagram represent elements of the posets. An arrow exists between the elements $x$ and $y$ if
Figure 6. An example of a Diagonal Flip

$x \leq y$ and the arrow points from $x$ to $y$ (See Figure 1 as an Example of a Hasse Diagram where the arrows are implied).

**Definition 3.14.** The *Bipartite Cambrian Polygon* is similar to the Tamari Polygon, but with a slight twist. Take $Q$ to be a convex $(n+2)$-gon with vertices labeled $0, 1, \cdots, n, n+1$. As before, we put the $0$ and $n+1$ vertices horizontally across from each other and connect them with a line. Now, we place the even vertices above the horizontal, and the odd vertices below the horizontal. We keep the condition that these new vertices increase in value when scanned from left to right. An example of these polygons can be seen in Figure 6.

**Definition 3.15.** The *Bipartite Cambrian Lattice* is the poset whose elements are the Bipartite Cambrian Polygons. A relationship between elements in the Bipartite Cambrian Lattice is given by diagonal flips. See Figure 1 for an example.

4. Commutation class of $c$-sortable words and maximum-length chains

4.1. Necessary mathematics for Cambrian Lattices. In this section, we introduce some definitions relevant for studying Cambrian lattices.

**Definition 4.1.** $S_n$ is the symmetric group of permutations of length $n$. $w_0$ is the reverse identity permutation in one-line notation. For example, in $S_4$, $w_0 = 4321$.

**Definition 4.2.** A *type $A_n$ Coxeter group* $W$ is generated by $S := \{s_1, \ldots, s_n\} \in W$ where $s_k^2 = 1$ for all $k \in [n]$. We have that $s_k s_k+1 s_k = s_{k+1} s_k s_{k+1}$ (called a *long braid relation*) and $s_k s_j = s_j s_k$ if $\|k - j\| \geq 2$ (called a *short braid relation*). The $s_1, \ldots, s_n$ are called simple reflections.

**Definition 4.3.** Every element $\pi \in W$ can be written (non-uniquely) as a word in the alphabet of $S$, that is, as a product of the simple reflections. $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}, s_{i_l} \in S = \{s_1, \ldots, s_n\}$. If $l$ is minimal among all words for $\pi$, then $l$ is called the length of $\pi$, and the word $s_{i_1} s_{i_2} \cdots s_{i_l}, s_{i_l}$ is called a reduced word for $\pi$.

**Remark 4.4.** Every element $\pi \in W$ can be written (non-uniquely) as a word in the alphabet of $S$, that is, as a product of the simple reflections. $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}, s_{i_l} \in S = \{s_1, \ldots, s_n\}$.

**Definition 4.5.** A *coxeter element* is a permutation of $c \in S_{n+1}$ which can be written as a product of $\{s_1, \ldots, s_n\}$. Given a coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \cdots a_n$ where $a_k \in \{s_1, \ldots, s_n\}$. Let $c^\infty = c c c = a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n \ldots$. Given $\pi \in S_{n+1}$, the $c$-sorting
word for $\pi$ (The $(a_1, a_2, \ldots, a_n)$-sorting word) is the subword of $c^{\infty}$ which is lexicographically first (as a sequence of positions in $c^{\infty}$) and is a reduced word for $\pi$.

**Definition 4.6.** Given a Coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \ldots a_n$ where $a_k \in \{s_1, \ldots, s_n\}$. Let $c^{\infty} = c c c = a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n \ldots$. Given $\pi \in S_{n+1}$, the $c$-sorting word for $\pi$ (The $(a_1, a_2, \ldots, a_n)$-sorting word) is the subword of $c^{\infty}$ which is lexicographically first (as a sequence of positions in $c^{\infty}$) and is a reduced word for $\pi$.

**Definition 4.7.** The $c$-Cambrian Lattice is the lattice created using all of the short braids on the Coxeter element $c$.

### 4.2. Commutation class of the $c$-sorting word of the longest element.

In this section, we review the fact that the maximum-length chains in a $c$-Cambrian lattice correspond to the reduced word in the commutation class of the $c$-sorting word for $w_0$.

**Lemma 4.8** ([8, Corollary 4.4]). The longest permutation $w_0$ of the symmetric group is is $c$-sortable for any $c$.

**Proposition 4.9** ([5, Proposition 2.5]). The $c$-singletons constitute a distributive sublattice of the (right) weak order on $S_n$.

The following proposition follows from [8] and [5]:

**Proposition 4.10.** Let $a = a_1 a_2 \ldots$ be a reduced word for a Coxeter element $c$. Let $w_0(a)$ be the $a$-sorting word of the longest element $w_0$ in $S_n$. Then the reduced words in the commutation class of $w_0(a)$ correspond to the maximum-length chains in the $c$-Cambrian lattice via the following bijection: Given a reduced word $u_1 u_2 \ldots u_{\binom{n}{2}}$ of $w_0$ in the commutation class of $w_0(a)$, the word $u_1 u_2 \ldots u_{\binom{n}{2}}$ is sent to the maximum-length chain

$$
\text{id} \overset{u_1}{\rightarrow} u_1 \overset{u_2}{\rightarrow} u_1 u_2 \overset{u_3}{\rightarrow} u_1 u_2 u_3 \ldots \overset{u_{\binom{n}{2}}}{\rightarrow} w_0
$$

### 4.3. The bipartite $c$-sorting words for the longest element.

In this section, we introduce the $c$-sorting words for the longest element in the Bipartite Cambrian Lattice. From here, it immediately follows using Proposition 4.10 that the commutation class of the $c$-sorting word can be used to find the maximum-length chains in the bipartite cambrian lattice.

**Lemma 4.11.** Consider permutations in $S_{n+1}$.

1. Let $c = s_1 s_3 \ldots s_2 s_4 \ldots$ be one of the bipartite Coxeter elements. The word $w_0(c)$ is the $c$-sorting word of $w_0$.
2. Similarly, if $c = s_2 s_4 \ldots s_1 s_3 \ldots$ is the other bipartite Coxeter element, the word $w_0(c)$ is the $c$-sorting word of $w_0$.

**Proof.** We explain part (1) of the lemma.

First, we show that the word $s_1 s_3 s_5 \ldots s_2 s_4 s_6 \ldots$ is equal to $w_0$.

We consider the result of applying this element to the identity permutation by examining the “trajectory” of an even number $2 \leq k \leq n + 1$. We refer to right-multiplication by $s_1 s_3 \cdots$ as an odd move and refer to right-multiplication by $s_2 s_4 \cdots$ as an even move. After either an odd or even move, $k$ moves one space to the left until it reaches the leftmost location:
After these \( k - 1 \) moves, we apply an even move, which does not affect \( k \). Then we apply odd and even moves until \( k \) reaches the \((n + 2 - k)\)th location.

For an example, set \( k = 4 \) and \( n = 7 \). The trajectory looks like:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{4} & \text{4} & \text{4} & \text{4} & \text{4} & \text{4} & \text{8} & \text{7} & \text{6} & \text{5} & \text{4} & \text{3} & \text{2} & \text{1}
\end{array}
\]

Odd elements preform a similar maneuver, where an odd \( k \) makes its way to the rightmost location before turning around and moving to the \((n + 2 - k)\)th location. Thus after a total of \( n + 1 \) odd or even moves, we obtain the reverse identity permutation.

The number of letters in this word is exactly the inversion number of \( w_0 \) so it is a reduced word for \( w_0 \). And it follows from the definition of c-sorting word that this word is the \((s_1 s_3 s_5 \ldots s_2 s_4 s_6 \ldots)\)-sorting word.

\[\square\]

**Proposition 4.12.** Let \( c = a_1 \ldots a_n \) be a reduced word for a bipartite Coxeter element. In the c-Cambrian lattice, the maximum-length chains correspond to the reduced word in the commutation class of the word given in Lemma 4.11.

**Proof.** This follows directly from Lemma 4.11 and Proposition 4.10. \[\square\]

5. **Largest collection of maximum-length chains which are disjoint**

In this section, we construct the largest collection of maximum-length chains which are disjoint except for the identity permutation and reverse identity permutation, \( w_0 \). To simplify the discussion we begin with several definitions.

**Definition 5.1.** We redefine the simple reflections \( s_k \) to be called *letters*. A letter is *odd* if \( k \) is odd, and *even* if \( k \) is even. The set of letters \( L_m \) contains the letters \( s_1, s_2, \ldots, s_{m-1}, s_m \)
Definition 5.2. A word is a string of letters. The set of words that contain only odd (resp. even) letters, with each letter only appearing once, is denoted by \( O \) (resp. \( E \)).

For example, \( w_1 = s_3s_1s_5 \in O \) and \( w_2 = s_2s_4s_6 \in E \).

A \( k \)-letter prefix is a word which contains only the first \( k \) letters of another word.

For example, the 3-letter prefix of the word \( s_1s_6s_4s_2s_7 \), written as \( P_3(s_1s_6s_4s_2s_7) \), is \( s_1s_6s_4 \).

Definition 5.3. Define the concatenation \( w_1 \cdot w_2 \) of two words \( w_1 \) and \( w_2 \) to be the word consisting of \( w_1 \) followed by \( w_2 \). For example, if \( w_1 = s_1s_3s_5 \) and \( w_2 = s_2s_4s_6 \), then \( w_1 \cdot w_2 = s_1s_3s_5s_2s_4s_6 \).

Definition 5.4. A cyclic shift is an action on a word completed by taking the first letter in a word and moving it towards the end of the word. For example, a cyclic shift on the word \( s_3s_1s_5 \) is \( s_1s_5s_3 \).

Definition 5.5. For any word \( w \), the garble of \( w \) is the set of all cyclic shifts of \( w \).

For example, the garble of \( s_1s_3s_5s_7 \) is \( \{ s_1s_3s_5s_7, s_3s_5s_7s_1, s_5s_7s_1s_3, s_7s_1s_3s_5 \} \).

Although it is not required in the definition, whenever we refer to the garble of a word \( w \) in the proof of Theorem 1.1, we will only refer to words which are either \( w = s_1s_3s_5 \cdots \in O \) or \( w = s_2s_4s_6 \cdots \in E \).

Definition 5.6. Two words \( w_1 \) and \( w_2 \) are considered to be vertex-disjoint if \( |w_1| = |w_2| \) and for all \( 1 \leq k < |w_1| \), the prefix \( P_k(w_1) \) consists of a different multiset of letters than does the prefix \( P_k(w_2) \).

Note that only proper prefixes are considered (i.e., not \( k = |w_1| \)). For example, \( w_1 = s_3s_1s_2s_3s_4s_5 \) is not vertex-disjoint with \( w_2 = s_1s_3s_2s_4s_5 \) because for \( k = 4 \), the multiset \( P_k(w_1) \), \( (\{s_3, s_1, s_2, s_3\}) \), is the same as the multiset \( P_k(w_2) \), \( (\{s_1, s_3, s_3, s_2\}) \). However, \( w_1 = s_3s_1s_2s_3s_4s_5 \) is vertex-disjoint with \( w_3 = s_5s_4s_3s_2s_1s_3 \).

Remark 5.7. For a word \( w \), if each letter in \( w \) appears only once (i.e., no repeats), then all words in the garble of \( w \) are pairwise vertex-disjoint. We later use the fact that for any \( w \in O \cup E \), the garble of \( w \) consists of vertex-disjoint words.

Definition 5.8. For two words \( w_1 \) and \( w_2 \), the boundary-crossing \( w_1 \leftrightarrow w_2 \) is defined to be the concatenation of \( w_1 \) with \( w_2 \), except with the final letter of \( w_1 \) swapped with the first letter of \( w_2 \). However, a boundary-crossing is only permitted if the two swapped letters have values which differ by at least 2 (i.e., the letters are \( s_i \) and \( s_j \) satisfying \( |i - j| \geq 2 \)).

For example, \( s_1s_3s_5s_7 \leftrightarrow s_2s_4s_6s_8 \) is the word \( s_1s_3s_5s_2s_7s_4s_6s_8 \).

Lemma 5.9. Consider the words \( x, x', y, y' \) where \( x \) and \( x' \) are vertex-disjoint and contain the same multisets of letters, and \( y \) and \( y' \) are vertex-disjoint and contain the same multisets of letters. Assume that \( x \) and \( y \) can perform a boundary-crossing, and \( x' \) and \( y' \) can perform a boundary-crossing. Then, the word \( x \leftrightarrow y \) is vertex-disjoint with \( x' \leftrightarrow y' \).

Proof. Let \( m = |x| = |x'| \). For \( 1 \leq r < m \), consider the prefix \( P_r(x \leftrightarrow y) \) and the prefix \( P_r(x' \leftrightarrow y') \).

Since \( r < m \), we have \( P_r(x \leftrightarrow y) = P_r(x) \) and \( P_r(x' \leftrightarrow y') = P_r(x') \). Since \( x \) and \( x' \) are vertex-disjoint, we know that the words \( x \leftrightarrow y \) and \( x' \leftrightarrow y' \) are also vertex-disjoint up to their \( m - 1 \)-th letters.

It also follows that \( P_m(x \leftrightarrow y) \) and \( P_m(x' \leftrightarrow y') \) are vertex disjoint. This is because the prefix \( P_m(x \leftrightarrow y) \) contains the first letter of \( y \), and the prefix \( P_m(x' \leftrightarrow y') \) contains the first letter of \( y' \).
These two letters differ by virtue of $y$ and $y'$ being vertex-disjoint, so it follows that the prefixes $P_r(x \leftrightarrow y)$ and $P_m(x' \leftrightarrow y')$ have different multisets of letters.

Lastly, we consider prefixes of length $m < r < |x \leftrightarrow y|$. In this case, both $P_r(x \leftrightarrow y)$ and $P_r(x' \leftrightarrow y')$ contain all of the letters from $x$ and all of the letters from $x'$, respectively. The prefixes $P_r(x \leftrightarrow y)$ and $P_r(x' \leftrightarrow y')$ also contain the first $r - m$ letters of $y$ and $y'$, respectively. By the assumption that $y$ and $y'$ are vertex disjoint, it follows that the prefixes $P_r(x \leftrightarrow y)$ and $P_r(x' \leftrightarrow y')$ contain different sets of letters than one-another. We have shown that for all $1 \leq r < |x \leftrightarrow y|$, the prefix $P_r(x \leftrightarrow y)$ consists of a different multiset of letters than does the prefix $P_r(x' \leftrightarrow y')$, thus completing the proof that $x \leftrightarrow y$ and $x' \leftrightarrow y'$ are vertex-disjoint. □

**Example 5.10.** Consider $x = s_1 s_3 s_5$, $x' = s_3 s_5 s_1$, $y = s_2 s_4 s_6$, $y' = s_4 s_6 s_2$. The previous lemma states that $x \leftrightarrow y = s_1 s_3 s_2 s_5 s_4 s_6$ is vertex disjoint with $x' \leftrightarrow y' = s_3 s_5 s_4 s_1 s_6 s_2$.

**Definition 5.11.** We define the *commutation class on chains* to be all of the chains that can be obtained by repeatedly swapping adjacent entries that are not adjacent valued. For example, the chains $s_1 s_3 s_5 s_2 s_4 s_6$ and $s_3 s_1 s_2 s_5 s_4 s_6$ are in the same commutation class because one chain is the same as the other but with $s_1$, $s_3$ swapped, and $s_5$, $s_2$ swapped.

It is the case that for each commutation class, we can take a base element and use it to derive every chain in the class.

**Definition 5.12.** For a positive integer $n$, we define the *Base Chain $B_n$* to be a chain that is constructed in the following way: $B_n$ is constructed with two repeating words. The first word $x \in O$ is defined to be all of the odd letters in $L_{n-1}$ in increasing order such that each odd letter appears once. The second word $y \in E$ strings together all of the even letters in $L_{n-1}$ in increasing order such that each even letter appears once. $B_n$ is the concatenation of these words $x y x y \cdots$ until the length of $B_n$ is $\binom{n}{2}$:

$$B_n = \underbrace{s_1 s_3 s_5 \cdots s_2 s_4 s_6 \cdots}_{\binom{n}{2}}$$

We will use this uniquely-constructed Base Chain ostensibly in the following section.

5.1. **The Swapping Technique.** Recall that our goal is to find the largest collection of maximum-length chains which are disjoint except for the identity and the reverse permutation, $w_0$. In this section, we will first solve an intermediate problem. In particular, our next goal will be to prove the following result: the maximum number of pairwise vertex-disjoint chains that can be found in the same commutation class as $B_n$ is $\lfloor \frac{n-1}{2} \rfloor$. We will later use this to prove Theorem 1.1. The Swapping Technique is a method introduced in this section which will aid in constructing these vertex-disjoint chains. The technique yields a construction of $\lfloor \frac{n-1}{2} \rfloor$ chains for all $n$ using the chain $B_n$, as denoted above.

We will now explain how the Swapping Technique works and follow with an example:

**Step 1**

The Swapping Technique first requires we start with $B_n$. We break apart $B_n$ in between each word $x$ and $y$. Then, we list all of the words in the same garble as $x$ underneath $x$ in a column, and do the same for $y$. As an example, let’s consider the chain $B_9$ (which is constructed from $x = s_1 s_3 s_5 s_7$ and
We restrict which words from the second column are allowed to be in a chain with words from the first column. This is done in the following way: A word from the second column is allowed to be in chain \(c_i\) if those two words can perform a boundary crossing. In other words, if the indices of the letters involved in the boundary-crossing differ by at least 2, then these two words are allowed to be in the same chain. The following lemma proves that it is feasible to build \(c_1, \ldots, c_{\left\lfloor \frac{n-1}{2} \right\rfloor}\) with this added constraint.

**Lemma 5.13.** It is possible to construct \(\left\lfloor \frac{n-1}{2} \right\rfloor\) chains \(c_1, \ldots, c_{\left\lfloor \frac{n-1}{2} \right\rfloor}\) such that for each chain, every word chosen to be in the chain from the \(i\)-th column can perform a boundary crossing with its predecessor word from the \(i-1\)-th column.

\[
y = s_2 s_8 s_4 s_6 s_8.
\]

Here’s what the breaking apart and listing steps look like for \(n = 9\):

\[
\begin{array}{cccccccccccc}
81 & 83 & 85 & 87 & 89 & 82 & 84 & 86 & 88 & 81 & 83 & 85 & 87 \\
87 & 81 & 83 & 85 & 87 & 89 & 82 & 84 & 86 & 88 & 81 & 83 & 85 & 87 \\
85 & 87 & 81 & 83 & 85 & 87 & 89 & 82 & 84 & 86 & 88 & 81 & 83 & 85 & 87 \\
83 & 85 & 87 & 81 & 83 & 85 & 87 & 89 & 82 & 84 & 86 & 88 & 81 & 83 & 85 & 87 \\
\end{array}
\]

By doing this process (for odd \(n\)), we have \(\left\lfloor \frac{n-1}{2} \right\rfloor\) words in each row of each column, and \(\binom{n}{2}\) letters in each full row. (For even \(n\), we have at least \(\left\lfloor \frac{n-1}{2} \right\rfloor\) words in each row of each column. This is because the even column will have \(\frac{n-1}{2}\) words in each row, but the odd column will have \(\frac{n-1}{2} + 1\) words in each row.) This is exactly the number of letters we need to create \(\left\lfloor \frac{n-1}{2} \right\rfloor\) chains all with length \(\binom{n}{2}\). The rest of the Swapping Technique involves strategically assigning all words into one of \(\left\lfloor \frac{n-1}{2} \right\rfloor\) chains such that every chain will have one word from each column. We build all chains simultaneously from left to right as follows:

**Step 2**
To begin constructing each chain, we start by considering just the first column of the \(B_n\) garbles. We trivially assign the first word in the column to be in chain \(c_1\), the second word in the column to be in chain \(c_2\), and so on until the last word in the column is in chain \(c_{\left\lfloor \frac{n-1}{2} \right\rfloor}\). Next, we decide which words from the second column will be in each chain \(c_1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\) with an added constraint: We restrict which words from the second column are allowed to be in a chain with words from the first column. This is done in the following way: A word from the second column is allowed to be in the same chain as a word in the first column if those two words can perform a boundary crossing. In other words, if the indices of the letters involved in the boundary-crossing differ by at least 2, then these two words are allowed to be in the same chain. The following lemma proves that it is feasible to build \(c_1, \ldots, c_{\left\lfloor \frac{n-1}{2} \right\rfloor}\) with this added constraint.
Proof. We prove Lemma 5.13 by explicitly constructing the chains \( c_1, \ldots, c_{\lfloor \frac{n-1}{2} \rfloor} \) using the Pairing Algorithm:

**Algorithm 1: Pairing Algorithm**

**Result:** This algorithm is used in the Swapping Technique to determine which words from each \( x \)-garble column and \( y \)-garble column are assigned to the same chain.

\[
\text{if } n \text{ is odd then}
\begin{align*}
\text{if } \lfloor \frac{n-1}{2} \rfloor \text{ odd then} \\
&\text{Case A: Put a word in column } i \text{ in the same chain as a word from column } i - 1 \text{ if the letters across the boundaries (the last letter of the word in column } i - 1 \text{ and the first letter of the word in column } i \text{) have values which are equal mod}(\lfloor \frac{n-1}{2} \rfloor). \\
\text{else} \\
&\text{Case B: Consider the letters involved in the boundary crossings of both columns } i \text{ and } i - 1. \text{ A word from either column is in set } X \text{ if its letter involved in the boundary crossing is in the set } \{s_1, \ldots, s_{\lfloor \frac{n-1}{2} \rfloor}\}. \text{ A word from either column is in set } Y \text{ otherwise. We pair a word in set } X \text{ with a word in set } Y \text{ if these words have letters involved in the boundary crossing whose values are equal mod}(\lfloor \frac{n-1}{2} \rfloor + 1). \\
\end{align*}
\]

**Note:** If we were to apply Case A in this scenario, then we realize that letters equal mod(\( \lfloor \frac{n-1}{2} \rfloor \)) have the same parity. Since we are constructing chains using words from columns with different parity, Case A cannot be applied.

\[
\text{end}
\]

\[
\text{else}
\begin{align*}
&\text{Remove the word in each column of } x \text{-garbles which starts with } s_1 \text{ and ends with } s_{n-1}. \\
&\text{if } \lfloor \frac{n-1}{2} \rfloor \text{ odd then} \\
&\quad \text{Repeat Case A} \\
&\text{else} \\
&\quad \text{Repeat Case B} \\
&\text{end}
\end{align*}
\]

This algorithm makes it such that every pair of words assigned to the same chain can perform a boundary crossing by assigning every last letter of a word in the chain to a permanent partner letter, which comes from the first letter of a word in the chain.

\[
\square
\]

With this restriction, we find a way to put each word from the second column into each of the \( \lfloor \frac{n-1}{2} \rfloor \) chains thanks to Lemma 5.13. We repeat this process for the rest of the columns, where a word in the \( i \)-th column is allowed to be in a chain with a word from the \( i - 1 \)-th column if these two words can perform a boundary crossing. If we follow this process and use the Pairing Algorithm for \( n = 9 \), we end up with four chains total, where words sharing the same color are in the same chain:

**Step 3**
There is one last step to the Swapping Technique. We can now take each chain in $c_1 \cdots c_{\lfloor \frac{n-1}{2} \rfloor}$ and perform boundary crossings across all odd and even words. By doing so, we transform our chains into our desired $\lfloor \frac{n-1}{2} \rfloor$ chains, $C_1 \cdots C_{\lfloor \frac{n-1}{2} \rfloor}$. For $n = 9$, the chains $C_1 \cdots C_4$ look like such:

\begin{align*}
C_1 &= s_1 s_3 s_5 s_7 s_8 s_9 s_2 s_4 s_6 s_8 s_4 s_5 s_7 s_5 s_8 s_6 s_8 s_5 s_2 s_7 s_5 s_1 s_3 s_2 s_4 s_1 s_6 s_3 s_5 s_8 s_7 s_6 s_8 s_5 s_3 s_1 s_6 s_5 s_3 s_8 s_5 s_7 s_8 \\
C_2 &= s_8 s_7 s_4 s_8 s_6 s_8 s_9 s_8 s_3 s_7 s_1 s_5 s_8 s_1 s_8 s_2 s_9 s_5 s_3 s_7 s_6 s_1 s_8 s_2 s_8 s_7 s_4 s_1 s_8 s_5 s_3 s_7 s_6 s_8 s_5 s_2 s_7 s_8 s_1 s_3 \\
C_3 &= s_5 s_8 s_1 s_8 s_9 s_2 s_4 s_9 s_6 s_3 s_5 s_8 s_7 s_8 s_5 s_9 s_8 s_5 s_2 s_8 s_3 s_7 s_1 s_8 s_3 s_5 s_4 s_7 s_8 s_8 s_5 s_2 s_7 s_8 s_1 s_3 \\
C_4 &= s_3 s_5 s_7 s_6 s_8 s_2 s_4 s_7 s_1 s_5 s_8 s_5 s_2 s_8 s_4 s_6 s_3 s_8 s_5 s_7 s_6 s_8 s_5 s_1 s_8 s_2 s_8 s_7 s_8 s_3 s_5 s_2 s_8 s_4 s_6 s_3 s_8 s_5 s_7 s_8 s_1
\end{align*}

Lemma 5.14. The Swapping Technique gives us $\lfloor \frac{n-1}{2} \rfloor$ chains that are in the same commutation class as $B_n$ and are all vertex disjoint.

Proof. We first realize that words in the same garble are in the same commutation class. This is because a cyclic shift action is equivalent to swapping the first letter with all of its predecessors until it reaches the end of the line. We also recognize that the boundary crossing moves just swaps elements. Therefore, since the words created from the Swapping Technique are just constructed from garbles from a concatenation of words in $B_n$ and from boundary crossings, we know from Definition 5.11 that these new words are in the same commutation class as $B_n$.

It remains to be proven that these chains are all pairwise vertex-disjoint. In order to show this, we can apply Lemma 5.9 repeatedly.

We know from Lemma 5.9 that $x \leftrightarrow y$ and $x' \leftrightarrow y'$ are vertex-disjoint. It then follows that $(x \leftrightarrow y) \leftrightarrow x$ and $(x' \leftrightarrow y') \leftrightarrow x'$ are vertex-disjoint. It also follows that $((x \leftrightarrow y) \leftrightarrow x) \leftrightarrow y$ and $((x' \leftrightarrow y') \leftrightarrow x') \leftrightarrow y'$ are vertex-disjoint. We can extend this logic to say that $x \leftrightarrow y \leftrightarrow x \leftrightarrow y \cdots$ and $x' \leftrightarrow y' \leftrightarrow x' \leftrightarrow y' \cdots$ are vertex-disjoint. Since any pair of chains from the Swapping Technique looks like $x \leftrightarrow y \leftrightarrow x \leftrightarrow y \cdots$ and $x' \leftrightarrow y' \leftrightarrow x' \leftrightarrow y' \cdots$, then we are done.

Lemma 5.15. If a set of chains in the commutation class of $B_n$ consists of more than $\lfloor \frac{n-1}{2} \rfloor$ elements, then at least two of the chains have a common letter.

Proof. Let $j = \lfloor \frac{n-1}{2} \rfloor$ and let $k = \binom{n}{2}$ be the number of letters in $B_n$. Assume toward contradiction that we have $m > j$ unique chains in the same commutation class as $B_n$ (each $k$ letters long), and assume that they are all pairwise vertex-disjoint. Since these $m$ chains are all in the same commutation class, then we know that they all must contain the same collection of letters, just ordered differently. Therefore, if they are all pairwise vertex-disjoint, then it must be true that they all must end with different letters (this is because if two chains both ended with the same letter, then the multisets of their $(k-1)$-letter prefixes would be equal, and therefore they would not be pairwise vertex-disjoint). To have a proof by contradiction, we will show that for even and odd $n$, all chains in the commutation class can only end with one of $j$ different letters.

First, we prove a contradiction by assuming $n$ to be even. In this case, we recall that the last $\lfloor \frac{n-1}{2} \rfloor$ letters of $B_n$ are $s_2, s_4, s_6, \ldots s_{n-2}$. If we were to apply swapping to $B_n$, all of these letters would be allowed to occupy the last spot in the chain of letters since they are all allowed to swap with each other. This means we are allowed to have at least $\lfloor \frac{n-1}{2} \rfloor$ chains which are all pairwise vertex-disjoint. Now, suppose we try to make $s_1$ – a letter not in the last $\lfloor \frac{n-1}{2} \rfloor$ letters of $B_n$ – the last letter. We know that in the process of creating more chains in the same commutation class as
$B_n$, $s_1$ and $s_2$ will never be allowed to swap places. But, $s_2$ comes after $s_1$ in the chain. This means it is impossible for $s_1$ to be allowed as the last letter in a chain in the same commutation class as $B_n$. We can say that $s_1$ is ”blocked” by $s_2$. We can generalize this for all $s_i$ for odd $i$. These $s_i$ will never be able to take the last spot in the chain because they will be ”blocked” by some $s_{i+1}$ which appears after $s_i$ in the chain.

Finally, we prove a contradiction by assuming $n$ to be odd. In this case, the last $\frac{n-1}{2}$ letters will be $s_1, s_3, s_5, \ldots s_{n-2}$. By similar arguments made above, these $\frac{n-1}{2}$ letters are the only letters allowed to be in the last position of the chain. And since $\frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$ for odd $n$, then this completes the proof.

We now describe the argument above for arbitrary values of $n$, and formalize it in the following theorem.

**Theorem 5.16.** For letters in $L_{n-1}$ (As defined in Definition 5.1), we can always find exactly $\lfloor \frac{n-1}{2} \rfloor$ chains in the equivalence class of $B_n$ that are vertex-disjoint.

**Proof.** We can construct $\lfloor \frac{n-1}{2} \rfloor$ chains using the Swapping Technique. The rest of the proof follows directly from Lemma 5.14, and Lemma 5.15. □

**Remark 5.17.** The Swapping Technique can be repeated but with a ”reverse” base chain that starts with even letters. It follows that the analysis of the Swapping Technique using the base chain directly applies to the reverse base chain. This remark is important for the proof of Theorem 1.1.

5.2. **Proof of Theorem 1.1.** In this subsection, we put the pieces together from Section 4 and Section 5 to prove Theorem 1.1.

The chain $B_n$ is actually equal to one of the bipartite coxeter element in Lemma 4.4 (the reverse base chain is equal to the other coxeter element). The word $w_0(B_n)$ is the $B_n$-sorting word of $w_0$ by construction. We see that the chains we get from $B_n$ using the Swapping Technique are actually the reduced words in the commutation class of $B_n$. This is because $B_n$ was constructed by using garbles and boundary swaps, which are all vertex-disjoint. If two words have the same letters, then vertex-disjoint words translates to the words being in the same commutation class. By Proposition 4.5, they are thus the maximum length chains of the Bipartite Cambrian Lattice.

It can be seen that these maximum length chains in the Bipartite Cambrian Lattice share only their first and last element, the identity and $w_0$. This is because maximum length chains that satisfy the vertex-disjoint definition only share their first and last elements by construction. Since they all share their first and last element, we just let the first chain claim the first and last element. The size of the other chains are thus equal and two less than the first chain. Thus, Theorem 5.16 therefore implies Theorem 1.1.

**References**


