TRIANGULATIONS OF CAMBRIAN LATTICES OF TYPE A

ROSE SILVER

Abstract. Dillworth's Theorem states that the maximal size of an antichain is equal to the minimal number of chains needed to cover the partially ordered set. We study the Greene-Kleitman partition of c-Cambrian lattices of type A. We partially compute the Greene-Kleitman partition of the Bipartite Cambrian Lattice. Professor Gordana Todorov is my consultant on the project.

Contents

1. Introduction

Dillworth's Theorem is one of the most groundbreaking discoveries in the field of combinatorics as it describes a universal property for all partially ordered sets (posets). The theorem states that the width of the largest antichain in a poset is equal to the minimum number of chains needed to cover the poset [2]. For now, if we arbitrarily let $\mu(\pi)$ represent the size of the largest antichain in the poset π, and $|\lambda(\pi)|$ be the minimum number of chains needed to cover a poset π, then

$$|\lambda(\pi)| = \mu(\pi).$$

The dual of Dillworth's theorem proven by Mirsky in [6] states that the size of the largest chain is equal to the minimum number of antichains needed to cover the poset (i.e., $\lambda(\pi) = |\mu(\pi)|$). The Greene-Kleitman Theorem provides a continuous transformation between the main and dual versions of Dilworth's theorem for a poset π [4]. Greene and Kleitman defined two invariants, $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$ and $\mu = (\mu_1, \mu_2, \cdots \mu_m)$, to describe weakly decreasing chain and antichain

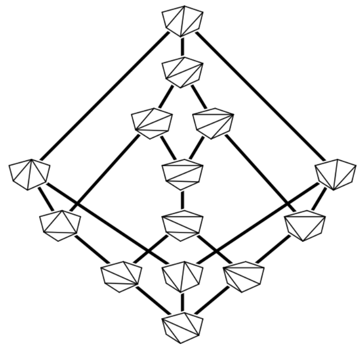


FIGURE 1. Bipartite Cambrian Lattice for $n = 4$

coverings respectively; see the definition in 3.2. The Greene-Kleitman theorem shows that these invariants are actually conjugate partitions and can be written as

$$\lambda_j = |\{k|\mu_k \geq j\}|,$$

$$\mu_j = |\{k|\lambda_k \geq j\}|.$$

This conjugacy is clear to see if λ and μ are represented with a Young Diagrams; see Figure 4 and Example 3.7. These results in total were very significant to the field of combinatorics as now we have tools to analyze and understand all partially ordered sets in detail.

One poset which is particularly interesting is the Tamari Lattice. Introduced by Tamari in [1], the most famous example of the Tamari Lattice is a partially ordered set where the elements represent binary bracketing of n objects. But the Tamari lattice has other realizations. This is because the number of elements for each n is the n-th Catalan number, and the Catalan number has over one hundred realizations [10]. The most relevant realization for this paper is of triangulations of $(n + 2)$-gons that look as in Figure 5.

In the past, some people have tried to apply the Greene-Kleitman invariant λ to the Tamari Lattice. In Early's PhD thesis, he has found the first three parts of the invariant for the Tamari Lattice [3]. Lim and Zhang followed suite and found the fourth and fifth parts of the Greene-Kleitman invariant for the Tamari Lattice [7].

Reading [9] generalized these Tamari triangulations of polygons to adjust for any $(n + 2)$-gon of choice (i.e., triangulations which happen if one adjusts the positions of vertices in the polygon). He calls these new posets Cambrian Lattice. This paper is interested in both the Tamari Lattice as well as one other of these Cambrian lattices called the Bipartite Cambrian Lattice. Here, the polygon has vertices as described in 3.14. For an example of the Bipartite Cambrian Lattice, see Figure 1.

The problem of this paper is to find as many parts of the Greene-Kleitman invariant as possible for both the Bipartite Cambrian Lattice and Tamari Lattice. This paper also seeks to find an understanding of these concepts from a representation-theoretic perspective as the triangulations of the polygons represents quivers of type A. Applications to this result would be contributing tools and techniques to other mathematicians for finding parts of the Greene-Kleitman invariant for the same or other posets. Also, another application is to further bridge the connection between the fields of algebraic combinatorics and representation theory and develop parallel meaning of objects in one field to the other. In this paper, we introduce the following theorem which finds the first $\lfloor \frac{n-1}{2} \rfloor$ parts of the Greene-Kleitman invariant.

Theorem 1.1 (Bipartite Theorem)**.** *Let $n \geq 4$. In the $s_1 s_3 \dots s_2 s_4 \dots$-Cambrian lattice and in the $s_2 s_4 \dots s_1 s_3 \dots$-Cambrian lattice, we have*

$$\lambda_1 - 2 = \lambda_2 = \lambda_3 = \ldots = \lambda_{\lfloor \frac{n-1}{2} \rfloor} > \lambda_{\lfloor \frac{n-1}{2} \rfloor + 1}.$$

The rest of the paper is organized as follows. In Section 2, we outline the problem being examined in this paper. In Section 3, we review some necessary mathematics needed to understand the problem statement in its entirety. In Section 4, we define commutation classes of c-sortable words and maximum length chains. Finally, in Section 5, we prove Theorem 1.1.

2. Problem Statement

The first goal of this paper is to find a lowerbound for the size of the largest antichain for the Tamari lattice. In other words, what is the minimum size of μ_1 for the Tamari Lattice for all n? In the past, researchers such as Early and Zhang [7] have found the first five parts of λ and have conjectured about finding more parts; see Figure 2. However, the width of the poset is not something which is known. This has been a challenging problem to solve, and the only well-known techniques for solving the largest antichain come from trying to find the largest antichain in the right-weak order of permutations.

A second goal of this paper is to find as many parts of λ for the Bipartite Cambrian Lattice for all n as possible. Finding parts of λ has not yet been extended into the Bipartite Cambrian Lattice, so this research would glean information in new mathematical territory. In order to find the parts of λ, it must be determined what these lattices look like for all n, something which has not been done before. It would be nice to have results similar to the ones in Figure 2.

The last goal is to study the Greene-Kleitman partition of Tamari lattices and other Cambrian lattices of type A, but from the approach from representation theory. In particular, there might be a connection between a special subposet in the Bipartite Cambrian Lattice and the Coxeter group. The subposet represents elements which might be in the longest chains of the Bipartite Lattice used to find the first several parts of λ, and the Coxeter group helps to uncover this subposet. For example, this subposet would be similar to the subposet represented by bold lines in Figure 2. If this connection is made, this might aid in understanding how the subposet works and if it can give any information to the largest parts of λ.

3. Necessary Mathematics

3.1. **Greene-Kleitman Theorem.** In this section, we review the necessary mathematics needed to understand the Greene-Kleitman Theorem which is directly related to the research problem.

Definition 3.1. A *partially ordered set* S is a set equipped with a binary relation $<$. For $x, y \in S$, we use $x < y$ to signify that x precedes y.

A *chain* in S is a sequence of elements $x_1, x_2, \ldots$ such that $x_1 < x_2 < \cdots$. An *antichain* in S is a subset A of S of elements such that every two distinct $x, y \in A$ are incomparable (i.e., neither $x < y$ nor $y < x$).

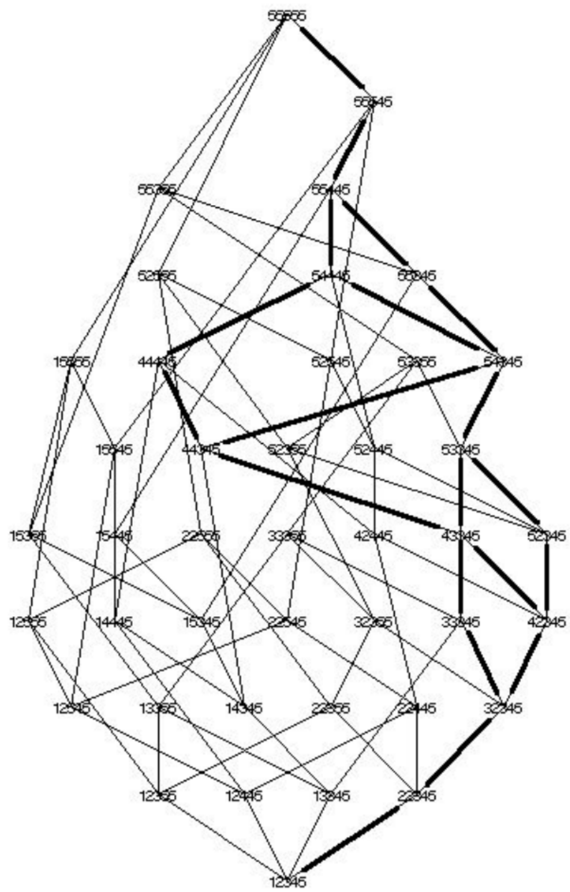


FIGURE 2. Tamari Lattice $n = 5$ by Early [3] used to compute parts of λ. This paper will attempt to find parts of λ for the Bipartite Cambrian Lattice in a similar way.

Definition 3.2. A *partition* of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of integers $\lambda_k \geq 0$ where the λ_k are weakly decreasing, that is,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$$

and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. The Greene-Kleitman Invariants are partitions, where n is the number of elements in the poset. The terms $\lambda_1, \lambda_2, \cdots, \lambda_l$ are called *parts*. If λ is a partition of n, we write $|\lambda| = n$.

With these definitions, we now introduce the Greene-Kleitman Theorem [4]:

Theorem 3.3 (Green-Kleitman Theorem)**.** *Let A_j denote the maximum size of j disjoint chains. Let B_j denote the maximum size of j disjoint antichains. Then A_j is equal to the number of k for which $B_k \geq j$, and B_j is equal to the number of k for which $A_k \geq j$.*

Using the notation from Theorem 3.3, let $\lambda_k = A_k - A_{k-1}$ where $A_0 = 0$, and let $\mu_k = B_k - B_{k-1}$. We define the *Greene-Kleitman invariants* $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$.

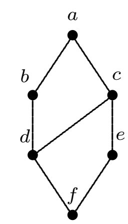


FIGURE 3. Partially ordered set P

Example 3.4. The poset in Figure 3 has its largest union of one chain (its largest chain) from $\{a, b, d, f\}$ and its largest union of two disjoint chains from $\{a, b, d, f\}$ and $\{c, e\}$. Therefore, $A_1 = 4$ and $A_2 = 6$. Computing λ, we have $\lambda = (4 - 0, 6 - 4) = (4, 2)$.

This poset has its largest union of one antichain (its largest antichain) from $\{b, c\}$, its largest union of two disjoint antichains from $\{b, c\}$ and $\{d, e\}$, its largest union of three disjoint antichains from $\{b, c\}$, $\{d, e\}$, and $\{f\}$, and finally its largest union of four disjoint antichains from $\{b, c\}$, $\{d, e\}$, $\{f\}$, and $\{a\}$. Therefore, $B_1 = 2, B_2 = 4, B_3 = 5$, and $B_4 = 6$. Therefore, $\mu = (2 - 0, 4 - 2, 5 - 4, 6 - 5) = (2, 2, 1, 1)$

Remark 3.5. In Example 3.4, it so happens that chains were chosen such that the set of the largest union of k chains was a subset of the set of the largest union of $k + 1$ chains. This is also true for the antichains in the example. However, this is not always the case for all posets. Likewise, there is not one unique choice of chains and antichains which compose the largest union. For example, one could choose to compose A_1 from the elements $\{a, d, e, f\}$.

Definition 3.6. The *Young Diagram* of a partition λ is an array of boxes having k left-justified rows with row i, where k is the number of parts of λ, containing λ_i boxes for $1 \leq i \leq k$.

The *conjugate partition* of λ is the partition μ whose shape is obtained from the Young Diagram of λ by interchanging rows and columns. Equivalently, the Young Diagram of μ is the reflection of the Young Diagram of λ about its main diagonal $y = -x$.

We now introduce an example which is helpful in understanding partially ordered sets, λ, μ, and their relationship to each other:

FIGURE 4. Young diagrams for $\lambda = \{4, 2\}$ and its conjugate partition $\mu = \{2, 2, 1, 1\}$

Example 3.7. Suppose we have $\lambda = (4, 2)$ and its conjugate $\mu = (2, 2, 1, 1)$ given by a poset such as in Figure 3. In Figure 4, we can represent λ and μ respectively with a Young Diagram. We can use the Young Diagram to check the different parts of λ and μ. As an example, we see that $\lambda_2 = 3 = |\{k|\mu_k \geq j\}|$.

3.2. **Triangulations of polygons.** Studying the values of A_j and B_j from Theorem 3.3 for specific posets is an especially interesting problem. In this section, we define two posets of special interest: the Tamari Lattice and the Bipartite Cambrian Lattice.

Definition 3.8. Take Q to be a convex $(n+2)$-gon with vertices labeled $0, 1, \cdots, n, n+1$. We put the 0 and $n+1$ vertices horizontally across from each other and connect them with a line. We put the remaining vertices between them, below the line, carving out a boat-like shape. We add the condition that these new vertices increase in value from left to right. This is the outline of the *Tamari Polygon*.

A *triangulation* of a polygon Q is a tiling of Q by triangles whose vertices are contained in the vertex set of Q [9].

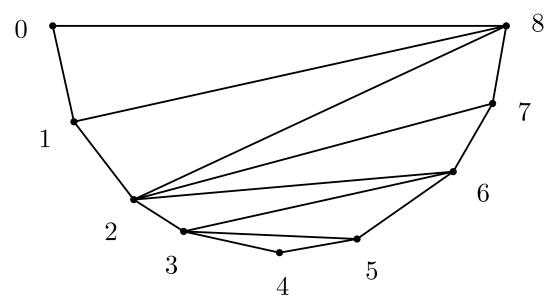


FIGURE 5. One Tamari Polygon for $n = 7$

Example 3.9. In Figure 5, we can see one example of a triangulation of a Tamari Polygon with $n = 7$. As we will later learn, these triangulations will make up the elements for the Tamari Lattice.

Definition 3.10. Let X denote the set of triangulations of an n-gon with vertex labels $1, 2, \ldots, n$. For any $x \in X$, a *diagonal flip* takes an edge (i, j) in the triangulation x, removes the edge to reveal a quadrilateral, and rearranges the edge to be a new edge (k, l) where k and l are the opposite corners of the quadrilateral. We call the edge involved in the diagonal flip the "flipped edge". A diagonal flip should preserve the fact that x is a triangulation. Two triangulations x and y which differ by one diagonal flip retain $x < y$ if the slope of the flipped edge in y is more positive than the slope of the flipped edge in x. Two triangulations x, z satisfy $x < z$ if z can be reached from x by a sequence of edge flips in an increasing order.

Example 3.11. In Figure 6, we see that the edge $(3, 6)$ has a negative slope, and gets "flipped" to $(2, 7)$ which has a positive slope.

Definition 3.12. The elements of the *Tamari Lattice* for a given n are all of the possible triangulations of this Tamari Polygon. A relationship between elements in the Tamari Lattice is given by diagonal flips.

Definition 3.13. A *Hasse Diagram* is a directed graph used to diagram posets. The nodes of the Hasse Diagram represent elements of the posets. An arrow exists between the elements x and y if

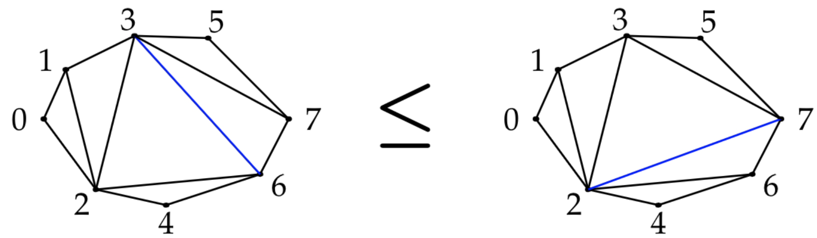


FIGURE 6. An example of a Diagonal Flip

$x \leq y$ and the arrow points from x to y (See Figure 1 as an Example of a Hasse Diagram where the arrows are implied).

Definition 3.14. The *Bipartite Cambrian Polygon* is similar to the Tamari Polygon, but with a slight twist. Take Q to be a convex $(n+2)$-gon with vertices labeled $0, 1, \cdots, n, n+1$. As before, we put the 0 and $n+1$ vertices horizontally across from each other and connect them with a line. Now, we place the even vertices above the horizontal, and the odd vertices below the horizontal. We keep the condition that these new vertices increase in value when scanned from left to right. An example of these polygons can be seen in Figure 6.

Definition 3.15. The *Bipartite Cambrian Lattice* is the poset whose elements are the Bipartite Cambrian Polygons. A relationship between elements in the Bipartite Cambrian Lattice is given by diagonal flips. See Figure 1 for an example.

4. Commutation class of c-sortable words and maximum-length chains

4.1. **Necessary mathematics for Cambrian Lattices.**

In this section, we introduce some definitions relevant for studying Cambrian lattices.

Definition 4.1. S_n is the symmetric group of permutations of length n. w_0 is the reverse identity permutation in one-line notation. For example, in S_4, $w_0 = 4321$.

Definition 4.2. A *type* A_n *Coxeter group* W is generated by $S := \{s_1, \ldots, s_n\} \in W$ where $s_k^2 = 1$ for all $k \in [n]$. We have that $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ (called a *long braid relation*) and $s_k s_j = s_j s_k$ if $\|k - j\| \geq 2$ (called a *short braid relation*). The $s_1, \ldots, s_n$ are called *simple reflections.*

Definition 4.3. Every element $\pi \in W$ can be written (non-uniquely) as a word in the alphabet of S, that is, as a product of the simple reflections. $\pi = s_{i_1} s_{i_2} \ldots s_{i_l}, s_{i_k} \in S = \{s_1, \ldots, s_n\}$. If l is minimal among all words for π, then l is called the *length* of π, and the word $s_{i_1} s_{i_2} \ldots s_{i_l}, s_{i_k}$ is called a *reduced word* for π.

Remark 4.4. Every element $\pi \in W$ can be written (non-uniquely) as a word in the alphabet of S, that is, as a product of the simple reflections. $\pi = s_{i_1} s_{i_2} \ldots s_{i_l}, s_{i_k} \in S = \{s_1, \ldots, s_n\}$.

Definition 4.5. A *coxeter element* is a permutation of $c \in S_{n+1}$ which can be written as a product of $\{s_1, \ldots s_n\}$. Given a coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \ldots a_n$ where $a_k \in \{s_1, \ldots, s_n\}$. Let $c^\infty = ccc = a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n \ldots$. Given $\pi \in S_{n+1}$, the *c-sorting* *word for* π (The $(a_n a_1 a_2 \dots a_n)$-sorting word) is the subword of c^{infty} which is lexicographically first (as a sequence of positions in c^∞) and is a reduced word for π.

Definition 4.6. Given a coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \dots a_n$ where $a_k \in \{s_1, \dots, s_n\}$. Let $c^\infty = ccc = a_1 a_2 \dots a_n a_1 a_2 \dots a_n a_1 a_2 \dots a_n \dots$. Given $\pi \in S_{n+1}$, the *c-sorting word for* π (The $(a_n a_1 a_2 \dots a_n)$-sorting word) is the subword of c^{infty} which is lexicographically first (as a sequence of positions in c^∞) and is a reduced word for π.

Definition 4.7. The c-Cambrian Lattice is the lattice created using all of the short braids on the coxeter element c.

4.2. **Commutation class of the c-sorting word of the longest element.** In this section, we review the fact that the maximum-length chains in a c-Cambrian lattice correspond to the reduced word in the commutation class of the c-sorting word for w_0.

Lemma 4.8 ([8, Corollary 4.4])**.** *The longest permutation w_0 of the symmetric group is is c-sortable for any c.*

Proposition 4.9 ([5, Proposition 2.5])**.** *The c-singletons constitute a distributive sublattice of the (right) weak order on S_n.*

The following proposition follows from [8] and [5]:

Proposition 4.10. *Let $a = a_1 a_2 \dots$ be a reduced word for a Coxeter element c. Let $w_0(a)$ be the a-sorting word of the longest element w_0 in S_n. Then the reduced words in the commutation class of $w_0(a)$ correspond to the maximum-length chains in the c-Cambrian lattice via the following bijection: Given a reduced word $u_1 u_2 \dots u_{\binom{n}{2}}$ of w_0 in the commutation class of $w_0(a)$, the word $u_1 u_2 \dots u_{\binom{n}{2}}$ is sent to the maximum-length chain*

$$id \xrightarrow{u_1} u_1 \xrightarrow{u_2} u_1 u_2 \xrightarrow{u_3} u_1 u_2 u_3 \dots \xrightarrow{u_{\binom{n}{2}}} w_0$$

4.3. **The bipartite c-sorting words for the longest element.** In this section, we introduce the c-sorting words for the longest element in the Bipartite Cambrian Lattice. From here, it immediately follows using Proposition 4.10 that the commutation class of the c-sorting word can be used to find the maximum-length chains in the bipartite cambrian lattice.

Lemma 4.11. *Consider permutations in S_{n+1}.*

(1) Let $c = s_1 s_3 \dots s_2 s_4 \dots$ be one of the bipartite Coxeter elements. The word $w_0(c)$ is the c-sorting word of w_0.
(2) Similarly, if $c = s_2 s_4 \dots s_1 s_3 \dots$ is the other bipartite Coxeter element, the word $w_0(c)$ is the c-sorting word of w_0.

Proof. We explain part (1) of the lemma.
First, we show that the word $s_1 s_3 s_5 \dots s_2 s_4 s_6 ...$ is equal to w_0.

We consider the result of applying this element to the identity permutation by examining the "trajectory" of an even number $2 \le k \le n+1$. We refer to right-multiplication by $s_1 s_3 \cdots$ as an *odd move* and refer to right-multiplication by $s_2 s_4 \cdots$ as an *even move*. After either an odd or even move, k moves one space to the left until it reaches the leftmost location:

$$\begin{array}{cccc} & & & k \\ & & k & \\ & \cdots & & \\ k & & & \end{array}$$

After these $k-1$ moves, we apply an even move, which does not affect k. Then we apply odd and even moves until k reaches the $(n+2-k)$th location.

$$\begin{array}{cccc} k & & & \\ & k & & \\ & & \cdots & \\ & & & k \end{array}$$

For an example, set $k=4$ and $n=7$. The trajectory looks like:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & & 4 & & & & & \\ & 4 & & & & & & \\ 4 & & & & & & & \\ 4 & & & & & & & \\ & 4 & & & & & & \\ & & 4 & & & & & \\ & & & 4 & & & & \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

Odd elements preform a similar maneuver, where an odd k makes its way to the rightmost location before turning around and moving to the $(n+2-k)$th location. Thus after a total of $n+1$ odd or even moves, we obtain the reverse identity permutation.

The number of letters in this word is exactly the inversion number of w_0 so it is a reduced word for w_0. And it follows from the definition of c-sorting word that this word is the $(s_1 s_3 s_5 \ldots s_2 s_4 s_6 \ldots)$-sorting word.

□

Proposition 4.12. *Let $c = a_1 \ldots a_n$ be a reduced word for a bipartite Coxeter element. In the c-Cambrian lattice, the maximum-length chains correspond to the reduced word in the commutation class of the word given in Lemma 4.11.*

Proof. This follows directly from Lemma 4.11 and Proposition 4.10. □

5. Largest collection of maximum-length chains which are disjoint

In this section, we construct the largest collection of maximum-length chains which are disjoint except for the identity permutation and reverse identity permutation, w_0. To simplify the discussion we begin with several definitions.

Definition 5.1. We redefine the simple reflections s_k to be called *letters*. A letter is *odd* if k is odd, and *even* if k is even. The set of letters L_m contains the letters $s_1, s_2, \cdots, s_{m-1}, s_m$

Definition 5.2. A *word* is a string of letters. The set of words that contain only odd (resp. even) letters, with each letter only appearing once, is denoted by O (resp. E).

For example, $w_1 = s_3s_1s_5 \in O$ and $w_2 = s_2s_4s_8 \in E$.

A *k-letter prefix* is a word which contains only the first k letters of another word.

For example, the 3-letter prefix of the word $s_1s_6s_4s_2s_7$, written as $P_3(s_1s_6s_4s_2s_7)$, is $s_1s_6s_4$.

Definition 5.3. Define the *concatenation* $w_1 \cdot w_2$ of two words w_1 and w_2 to be the word consisting of w_1 followed by w_2. For example, if $w_1 = s_1s_3s_5$ and $w_2 = s_2s_4s_6$, then $w_1 \cdot w_2 = s_1s_3s_5s_2s_4s_6$.

Definition 5.4. A *cyclic shift* is an action on a word completed by taking the first letter in a word and moving it towards the end of the word. For example, a cyclic shift on the word $s_3s_1s_5$ is $s_1s_5s_3$.

Definition 5.5. For any word w, the *garble* of w is the set of all cyclic shifts of a w.

For example, the garble of $s_1s_3s_5s_7$ is $\{s_1s_3s_5s_7, s_3s_5s_7s_1, s_5s_7s_1s_3, s_7s_1s_3s_5\}$.

Although it is not required in the definition, whenever we refer to the garble of a word w in the proof of Theorem 1.1, we will only refer to words which are either $w = s_1s_3s_5 \cdots \in O$ or $w = s_2s_4s_6 \cdots \in E$.

Definition 5.6. Two words w_1 and w_2 are considered to be *vertex-disjoint* if $|w_1| = |w_2|$ and for all $1 \leq k < |w_1|$, the prefix $P_k(w_1)$ consists of a different multiset of letters than does the prefix $P_k(w_2)$. Note that only proper prefixes are considered (i.e., not $k = |w_1|$). For Example, $w_1 = s_3s_1s_2s_3s_4s_5$ is not vertex-disjoint with $w_2 = s_1s_3s_3s_2s_4s_5$ because for $k = 4$, the multiset $P_4(w_1)$, $(\{s_3, s_1, s_2, s_3\})$, is the same as the multiset $P_4(w_2)$, $(\{s_1, s_3, s_3, s_2\})$. However, $w_1 = s_3s_1s_2s_3s_4s_5$ is vertex-disjoint with $w_3 = s_5s_4s_3s_2s_1s_3$.

Remark 5.7. For a word w, if each letter in w appears only once (i.e., no repeats), then all words in the garble of w are pairwise vertex-disjoint. We later use the fact that for any $w \in O \cup E$, the garble of w consists of vertex-disjoint words.

Definition 5.8. For two words w_1 and w_2, the *boundary-crossing* $w_1 \Leftrightarrow w_2$ is defined to be the concatenation of w_1 with w_2, except with the final letter of w_1 swapped with the first letter of w_2. However, a boundary-crossing is only permitted if the two swapped letters have values which differ by at least 2 (i.e., the letters are s_i and s_j satisfying $|i - j| \geq 2$).

For example, $s_1s_3s_5s_7 \Leftrightarrow s_2s_4s_6s_8$ is the word $s_1s_3s_5s_2s_7s_4s_6s_8$.

Lemma 5.9. *Consider the words x, x', y, y' where x and x' are vertex-disjoint and contain the same multisets of letters, and y and y' are vertex-disjoint and contain the same multisets of letters. Assume that x and y can perform a boundary-crossing, and x' and y' can perform a boundary-crossing. Then, the word $x \Leftrightarrow y$ is vertex-disjoint with $x' \Leftrightarrow y'$.*

Proof. Let $m = |x| = |x'|$. For $1 \leq r < m$, consider the prefix $P_r(x \Leftrightarrow y)$ and the prefix $P_r(x' \Leftrightarrow y')$. Since $r < m$, we have $P_r(x \Leftrightarrow y) = P_r(x)$ and $P_r(x' \Leftrightarrow y') = P_r(x')$. Since x and x' are vertex-disjoint, we know that the words $x \Leftrightarrow y$ and $x' \Leftrightarrow y'$ are also vertex-disjoint up to their $m - 1$-th letters.

It also follows that $P_m(x \Leftrightarrow y)$ and $P_m(x' \Leftrightarrow y')$ are vertex disjoint. This is because the prefix $P_m(x \Leftrightarrow y)$ contains the first letter of y, and the prefix $P_m(x' \Leftrightarrow y')$ contains the first letter of y'.

These two letters differ by virtue of y and y' being vertex-disjoint, so it follows that the prefixes $P_m(x \Leftrightarrow y)$ and $P_m(x' \Leftrightarrow y')$ have different multisets of letters.

Lastly, we consider prefixes of length $m < r < |x \Leftrightarrow y|$. In this case, both $P_r(x \Leftrightarrow y)$ and $P_r(x' \Leftrightarrow y')$ contain all of the letters from x and all of the letters from x', respectively. The prefixes $P_r(x \Leftrightarrow y)$ and $P_r(x' \Leftrightarrow y')$ also contain the first $r - m$ letters of y and y', respectively. By the assumption that y and y' are vertex disjoint, it follows that the prefixes $P_r(x \Leftrightarrow y)$ and $P_r(x' \Leftrightarrow y')$ contain different sets of letters than one-another. We have shown that for all $1 \le r < |x \Leftrightarrow y|$, the prefix $P_r(x \Leftrightarrow y)$ consists of a different multiset of letters than does the prefix $P_r(x' \Leftrightarrow y')$, thus completing the proof that $x \Leftrightarrow y$ and $x' \Leftrightarrow y'$ are vertex-disjoint. □

Example 5.10. Consider $x = s_1s_3s_5$, $x' = s_3s_5s_1$, $y = s_2s_4s_6$, $y' = s_4s_6s_2$. The previous lemma states that $x \Leftrightarrow y = s_1s_3s_2s_5s_4s_6$ is vertex disjoint with $x' \Leftrightarrow y' = s_3s_5s_4s_1s_6s_2$.

Definition 5.11. We define the *commutation class on chains* to be all of the chains that can be obtained by repeatedly swapping adjacent entries that are not adjacently valued. For example, the chains $s_1s_3s_5s_2s_4s_6$ and $s_3s_1s_2s_5s_4s_6$ are in the same commutation class because one chain is the same as the other but with s_1, s_3 swapped, and s_5, s_2 swapped.

It is the case that for each commutation class, we can take a base element and use it to derive every chain in the class.

Definition 5.12. For a positive integer n, we define the *Base Chain* B_n to be a chain that is constructed in the following way: B_n is constructed with two repeating words. The first word $x \in O$ is defined to be all of the odd letters in L_{n-1} in increasing order such that each odd letter appears once. The second word $y \in E$ strings together all of the even letters in L_{n-1} in increasing order such that each even letter appears once. B_n is the concatenation of these words $xyxy\cdots$ until the length of B_n is $\binom{n}{2}$:

$$B_n = \underbrace{s_1s_3s_5\cdots s_2s_4s_6\cdots s_1s_3s_5\cdots}_{\binom{n}{2}}$$

We will use this uniquely-constructed Base Chain ostensibly in the following section.

5.1. **The Swapping Technique.** Recall that our goal is to find the largest collection of maximum-length chains which are disjoint except for the identity and the reverse permutation, w_0. In this section, we will first solve an intermediate problem. In particular, our next goal will be to prove the following result: the maximum number of pairwise vertex-disjoint chains that can be found in the same commutation class as B_n is $\lfloor \frac{n-1}{2} \rfloor$. We will later use this to prove Theorem 1.1. The Swapping Technique is a method introduced in this section which will aid in constructing these vertex-disjoint chains. The technique yields a construction of $\lfloor \frac{n-1}{2} \rfloor$ chains for all n using the chain B_n, as denoted above.

We will now explain how the Swapping Technique works and follow with an example:

Step 1

The Swapping Technique first requires we start with B_n. We break apart B_n in between each word x and y. Then, we list all of the words in the same garble as x underneath x in a column, and do the same for y. As an example, let's consider the chain B_9 (which is constructed from $x = s_1s_3s_5s_7$ and

$y = s_2s_4s_6s_8$).

$$s_1s_3s_5s_7s_2s_4s_6s_8s_1s_3s_5s_7s_2s_4s_6s_8s_1s_3s_5s_7s_2s_4s_6s_8s_1s_3s_5s_7s_2s_4s_6s_8s_1s_3s_5s_7$$

Here's what the breaking apart and listing steps look like for $n = 9$:

$s_1s_3s_5s_7$	$s_8s_2s_4s_6$	$s_1s_3s_5s_7$	$s_8s_2s_4s_6$	$s_1s_3s_5s_7$	$s_8s_2s_4s_6$	$s_1s_3s_5s_7$	$s_8s_2s_4s_6$	$s_1s_3s_5s_7$
$s_7s_1s_3s_5$	$s_6s_8s_2s_4$	$s_7s_1s_3s_5$	$s_6s_8s_2s_4$	$s_7s_1s_3s_5$	$s_6s_8s_2s_4$	$s_7s_1s_3s_5$	$s_6s_8s_2s_4$	$s_7s_1s_3s_5$
$s_5s_7s_1s_3$	$s_4s_6s_8s_2$	$s_5s_7s_1s_3$	$s_4s_6s_8s_2$	$s_5s_7s_1s_3$	$s_4s_6s_8s_2$	$s_5s_7s_1s_3$	$s_4s_6s_8s_2$	$s_5s_7s_1s_3$
$s_3s_5s_7s_1$	$s_2s_4s_6s_8$	$s_3s_5s_7s_1$	$s_2s_4s_6s_8$	$s_3s_5s_7s_1$	$s_2s_4s_6s_8$	$s_3s_5s_7s_1$	$s_2s_4s_6s_8$	$s_3s_5s_7s_1$

By doing this process (for odd n), we have $\lfloor \frac{n-1}{2} \rfloor$ words in each row of each column, and $\binom{n}{2}$ letters in each full row. (For even n, we have at least $\lfloor \frac{n-1}{2} \rfloor$ words in each row of each column. This is because the even column will have $\frac{n-1}{2}$ words in each row, but the odd column will have $\frac{n-1}{2} + 1$ words in each row.) This is exactly the number of letters we need to create $\lfloor \frac{n-1}{2} \rfloor$ chains all with length $\binom{n}{2}$. The rest of the Swapping Technique involves strategically assigning all words into one of $\lfloor \frac{n-1}{2} \rfloor$ chains such that every chain will have one word from each column. We build all chains simultaneously from left to right as follows:

Step 2

To begin constructing each chain, we start by considering just the first column of the B_n garbles. We trivially assign the first word in the column to be in chain c_1, the second word in the column to be in chain c_2, and so on until the last word in the column is in chain $c_{\lfloor \frac{n-1}{2} \rfloor}$. Next, we decide which words from the second column will be in each chain $c_1, \ldots, c_{\lfloor \frac{n-1}{2} \rfloor}$ with an added constraint: We restrict which words from the second column are allowed to be in a chain with words from the first column. This is done in the following way: A word from the second column is allowed to be in the same chain as a word in the first column if those two words can perform a boundary crossing. In other words, if the indices of the letters involved in the boundary-crossing differ by at least 2, then these two words are allowed to be in the same chain. The following lemma proves that it is feasible to build $c_1, \ldots, c_{\lfloor \frac{n-1}{2} \rfloor}$ with this added constraint.

Lemma 5.13. *It is possible to construct $\lfloor \frac{n-1}{2} \rfloor$ chains $c_1, \ldots, c_{\lfloor \frac{n-1}{2} \rfloor}$ such that for each chain, every word chosen to be in the chain from the i-th column can perform a boundary crossing with its predecessor word from the $i-1$-th column.*

Proof. We prove Lemma 5.13 by explicitly constructing the chains $c_1, \ldots, c_{\lfloor \frac{n-1}{2} \rfloor}$ using the Pairing Algorithm:

Algorithm 1: Pairing Algorithm

Result: This algorithm is used in the Swapping Technique to determine which words from each x-garble column and y-garble column are assigned to the same chain.

if n *is odd* **then**

  if $\lfloor \frac{n-1}{2} \rfloor$ *odd* **then**

    Case A: Put a word in column i in the same chain as a word from column $i-1$ if the letters across the boundaries (the last letter of the word in column $i-1$ and the first letter of the word in column i) have values which are equal $mod(\lfloor \frac{n-1}{2} \rfloor)$.

  else

    Case B: Consider the letters involved in the boundary crossings of both columns i and $i-1$. A word from either column is in set X if its letter involved in the boundary crossing is in the set $\{s_1, \ldots, s_{\lfloor \frac{n-1}{2} \rfloor}\}$. A word from either column is in set Y otherwise. We pair a word in set X with a word in set Y if these words have letters involved in the boundary crossing whose values are equal $mod(\lfloor \frac{n-1}{2} \rfloor + 1)$.

    Note: If we were to apply Case A in this scenario, then we realize that letters equal $mod(\lfloor \frac{n-1}{2} \rfloor)$ have the same parity. Since we are constructing chains using words from columns with different parity, Case A cannot be applied.

  end

else

  Remove the word in each column of x-garbles which starts with s_1 and ends with s_{n-1}.

  if $\lfloor \binom{n-1}{2} \rfloor$ *odd* **then**

    Repeat Case A

  else

    Repeat Case B

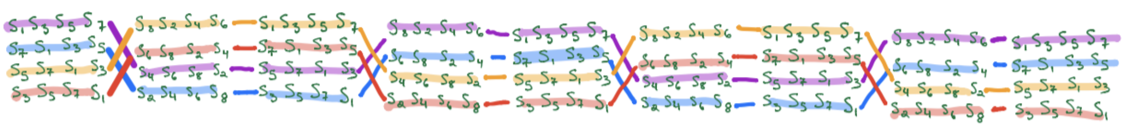
  end

end

This algorithm makes it such that every pair of words assigned to the same chain can perform a boundary crossing by assigning every last letter of a word in the chain to a permanent partner letter, which comes from the first letter of a word in the chain.

□

With this restriction, we find a way to put each word from the second column into each of the $\lfloor \frac{n-1}{2} \rfloor$ chains thanks to Lemma 5.13. We repeat this process for the rest of the columns, where a word in the i-th column is allowed to be in a chain with a word from the $i-1$-th column if these two words can perform a boundary crossing. If we follow this process and use the Pairing Algorithm for $n = 9$, we end up with four chains total, where words sharing the same color are in the same chain:



Step 3

There is one last step to the Swapping Technique. We can now take each chain in $c_1 \cdots c_{\lfloor \frac{n-1}{2} \rfloor}$ and perform boundary crossings across all odd and even words. By doing so, we transform our chains into our desired $\lfloor \frac{n-1}{2} \rfloor$ chains, $C_1 \cdots C_{\lfloor \frac{n-1}{2} \rfloor}$. For $n = 9$, the chains $C_1 \cdots C_4$ look like such:

$$C_1 = s_1 s_3 s_5 s_4 s_7 s_6 s_8 s_5 s_2 s_7 s_1 s_8 s_3 s_2 s_4 s_1 s_6 s_3 s_5 s_4 s_7 s_6 s_8 s_5 s_2 s_7 s_1 s_8 s_3 s_2 s_4 s_1 s_6 s_3 s_5 s_7$$
$$C_2 = s_7 s_1 s_3 s_2 s_5 s_4 s_6 s_3 s_8 s_5 s_7 s_6 s_1 s_8 s_2 s_7 s_4 s_1 s_3 s_2 s_5 s_4 s_6 s_3 s_8 s_5 s_7 s_6 s_1 s_8 s_2 s_7 s_4 s_1 s_3 s_5$$
$$C_3 = s_5 s_7 s_1 s_8 s_3 s_2 s_4 s_1 s_6 s_3 s_5 s_4 s_7 s_6 s_8 s_5 s_2 s_7 s_1 s_8 s_3 s_2 s_4 s_1 s_6 s_3 s_5 s_4 s_7 s_6 s_8 s_5 s_2 s_7 s_1 s_3$$
$$C_4 = s_3 s_5 s_7 s_6 s_1 s_8 s_2 s_4 s_7 s_1 s_3 s_5 s_2 s_4 s_6 s_3 s_8 s_5 s_7 s_6 s_1 s_8 s_2 s_4 s_7 s_1 s_3 s_5 s_2 s_4 s_6 s_3 s_8 s_5 s_7 s_1$$

Lemma 5.14. *The Swapping Technique gives us $\lfloor \frac{n-1}{2} \rfloor$ chains that are in the same commutation class as B_n and are all vertex disjoint.*

Proof. We first realize that words in the same garble are in the same commutation class. This is because a cyclic shift action is equivalent to swapping the first letter with all of its predecessors until it reaches the end of the line. We also recognize that the boundary crossing moves just swaps elements. Therefore, since the words created from the Swapping Technique are just constructed from garbles from a concatenation of words in B_n and from boundary crossings, we know from Definition 5.11 that these new words are in the same commutation class as B_n.

It remains to be proven that these chains are all pairwise vertex-disjoint. In order to show this, we can apply Lemma 5.9 repeatedly.

We know from Lemma 5.9 that $x \Leftrightarrow y$ and $x' \Leftrightarrow y'$ are vertex-disjoint. It then follows that $(x \Leftrightarrow y) \Leftrightarrow x$ and $(x' \Leftrightarrow y') \Leftrightarrow x'$ are vertex-disjoint. It also follows that $((x \Leftrightarrow y) \Leftrightarrow x) \Leftrightarrow y)$ and $((x' \Leftrightarrow y') \Leftrightarrow x') \Leftrightarrow y')$ are vertex-disjoint. We can extend this logic to say that $x \Leftrightarrow y \Leftrightarrow x \Leftrightarrow y \cdots$ and $x' \Leftrightarrow y' \Leftrightarrow x' \Leftrightarrow y' \cdots$ are vertex-disjoint. Since any pair of chains from the Swapping Technique looks like $x \Leftrightarrow y \Leftrightarrow x \Leftrightarrow y \cdots$ and $x' \Leftrightarrow y' \Leftrightarrow x' \Leftrightarrow y' \cdots$, then we are done.

□

Lemma 5.15. *If a set of chains in the commutation class of B_n consists of more than $\lfloor \frac{n-1}{2} \rfloor$ elements, then at least two of the chains have a common letter.*

Proof. Let $j = \lfloor \frac{n-1}{2} \rfloor$ and let $k = \binom{n}{2}$ be the number of letters in B_n. Assume toward contradiction that we have $m > j$ unique chains in the same commutation class as B_n (each k letters long), and assume that they are all pairwise vertex-disjoint. Since these m chains are all in the same commutation class, then we know that they all must contain the same collection of letters, just ordered differently. Therefore, if they are all pairwise vertex-disjoint, then it must be true that they all must end with different letters (this is because if two chains both ended with the same letter, then the multisets of their $(k-1)$-letter prefixes would be equal, and therefore they would not be pairwise vertex-disjoint). To have a proof by contradiction, we will show that for even and odd n, all chains in the commutation class can only end with one of j different letters.

First, we prove a contradiction by assuming n to be even. In this case, we recall that the last $\lfloor \frac{n-1}{2} \rfloor$ letters of B_n are $s_2, s_4, s_6, \ldots s_{n-2}$. If we were to apply swapping to B_n, all of these letters would be allowed to occupy the last spot in the chain of letters since they are all allowed to swap with each other. This means we are allowed to have at least $\lfloor \frac{n-1}{2} \rfloor$ chains which are all pairwise vertex-disjoint. Now, suppose we try to make s_1 – a letter not in the last $\lfloor \frac{n-1}{2} \rfloor$ letters of B_n – the last letter. We know that in the process of creating more chains in the same commutation class as

B_n, s_1 and s_2 will never be allowed to swap places. But, s_2 comes after s_1 in the chain. This means it is impossible for s_1 to be allowed as the last letter in a chain in the same commutation class as B_n. We can say that s_1 is "blocked" by s_2. We can generalize this for all s_i for odd i. These s_i will never be able to take the last spot in the chain because they will be "blocked" by some s_{i+1} which appears after s_i in the chain.

Finally, we prove a contradiction by assuming n to be odd. In this case, the last $\frac{n-1}{2}$ letters will be $s_1, s_3, s_5, \ldots s_{n-2}$. By similar arguments made above, these $\frac{n-1}{2}$ letters are the only letters allowed to be in the last position of the chain. And since $\frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$ for odd n, then this completes the proof. □

We now describe the argument above for arbitrary values of n, and formalize it in the following theorem.

Theorem 5.16. *For letters in L_{n-1} (As defined in Definition 5.1), we can always find exactly $\lfloor \frac{n-1}{2} \rfloor$ chains in the equivalence class of B_n that are vertex-disjoint.*

Proof. We can construct $\lfloor \frac{n-1}{2} \rfloor$ chains using the Swapping Technique. The rest of the proof follows directly from Lemma 5.14, and Lemma 5.15.

□

Remark 5.17. The Swapping Technique can be repeated but with a "reverse" base chain that starts with even letters. It follows that the analysis of the Swapping Technique using the base chain directly applies to the reverse base chain. This remark is important for the proof of Theorem 1.1.

5.2. **Proof of Theorem 1.1.** In this subsection, we put the pieces together from Section 4 and Section 5 to prove Theorem 1.1.

The chain B_n is actually equal to one of the bipartite coxeter element in Lemma 4.4 (the reverse base chain is equal to the other coxeter element). The word $w_0(B_n)$ is the B_n-sorting word of w_0 by construction. We see that the chains we get from B_n using the Swapping Technique are actually the reduced words in the commutation class of B_n. This is because B_n was constructed by using garbles and boundary swaps, which are all vertex-disjoint. If two words have the same letters, then vertex-disjoint words translates to the words being in the same commutation class. By Proposition 4.5, they are thus the maximum length chains of the Bipartite Cambrian Lattice.

It can be seen that these maximum length chains in the Bipartite Cambrian Lattice share only their first and last element, the identity and w_0. This is because maximum length chains that satisfy the vertex-disjoint definition only share their first and last elements by construction. Since they all share their first and last element, we just let the first chain claim the first and last element. The size of the other chains are thus equal and two less than the first chain. Thus, Theorem 5.16 therefore implies Theorem 1.1.

References

[1] The algebra of bracketings and their enumeration. *Nieuw Archief voor Wiskunde. Derde Serie*, 10:131–146, 1962.

[2] Robert Dilworth. A decomposition theorem for partially ordered sets. *Annals of mathematics.*, 51(1):161–166, 1950.

[3] Edward Early. Chain lengths in the tamari lattice. *Annals of Combinatorics*, 8(1):37–43, 2004.

[4] Curtis Greene and Daniel J Kleitman. The structure of sperner k-families. *Journal of Combinatorial Theory, Series A*, 20(1):41 – 68, 1976.

[5] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas. Permutahedra and generalized associahedra. *Adv. Math.*, 226(1):608–640, 2011.

[6] Leon Mirsky. A dual of dilworth's decomposition theorem. *The American mathematical monthly.*, 78(8):876–877, 1971.

[7] Mark Zhang Raphael-Joel Lim. Previously unknown parts of the greene-kleitman partition for the tamari lattice, 2008.

[8] Nathan Reading. Clusters, Coxeter-sortable elements and noncrossing partitions. *Trans. Amer. Math. Soc.*, 359(12):5931–5958, 2007.

[9] Nathan Reading. From the Tamari lattice to Cambrian lattices and beyond. In *Associahedra, Tamari lattices and related structures*, volume 299 of *Prog. Math. Phys.*, pages 293–322. Birkhäuser/Springer, Basel, 2012.

[10] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.